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THE COMPARATIVE ANALYSIS OF REGULARIZATION METHODS APPLICATION FOR THE PROCESSING OF INCOSISTENT EXPERT EVALUATIONS IN EDUCATION

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ABSTRACT

Present paper is focused on the study of stability issues for some "traditional" models aimed at analysis of expert evaluations. It is demonstrated that estimation of the complex indicator true value for each student in the space of characteristics is dependent on the stable inverted transformation of the initial data matrix, and which is commonly regarded to be an ill-conditioned matrix: for obtaining of regularized solution it is considered a classical Tikhonov regularization method applying the traditional approaches to the optimal regularization parameter selection. There are shown the shortcomings of those traditional approaches, and proposed a principal new approach to determination of optimal regularization parameter. For finding the residual (as well as for the obtained optimal regularization parameter) between the normal pseudosolution and solution based on the developed method there are obtained the upper estimates, and based on the obtained evaluation it is proved both convergence property of the found regularized solution to the normal pseudosolution and the fact that proposed method is inducing the Tikhonov regularizing operator.

Key Words: Mathematical model, expert assessments, objectivity, coherence, Tikhonov regularization method, normal pseudosolution, regularized solution.

INTRODUCTION

One of the major problems in analyzing experts' data that require quantitative assessment methods is the problem of adequate mathematical tools (models and methods) selection for the expression of experts' opinions and further processing of the information obtained on the basis of operator-algebraic and/or statistical approaches. The purpose of these diverse studies is to determine the objectivity of experts' evaluations at the decision-making process and while constructing the integral indicators. In fact, an objective analysis of the environmental, economic, socio-political and socio-psychological, educational, sports, etc. systems depends on the overall address of the problem – the problem of determining the objectivity of experts' evaluation. Currently, there are a lot of models, approaches and methods for planning expert opinion polls, as well as collecting, processing and analysis of experts' opinions. If to omit the details relevant to the existing models and methods, they can be conventionally regrouped into two "larger" classes: 1) probabilistic-statistical-parametric models, including a probability of various assumptions (e.g., the assumption of normality of the experts assessments distribution), which actually are not sufficiently justified, 2) operator (linear or non-linear) deterministic models, which, first, are unstable (i.e. have an increased sensitivity degree of their solutions to the possible perturbations of initial data, even to arbitrarily small ones) and, secondly, are not enough algorithmized. More information about these two classes of models and methods of experts' evaluations analysis can be obtained from, for example, (van der Linden & Hambleton, 1997; Neiman & Hlebnikov; Litvak & Tyurin, 1979; Zagoruyko, 2002; Cherepanov, 1989), in which there is a vast bibliography on this subject. Main generality of mathematical models of these two classes is rooted in two aspects on which basis mathematical models are constructed:

- *The concept of experts' opinions consistency*, when a feasible solution, including the optimal solution, madeon the basis of correlated experts' opinion, i.e. there are excluded / declined from the panel of experts those experts whose views differ from the opinions of most experts in the commission. As it has been justified in (Guseynov & Berezhnoy, 2011), this approach to the acceptance of a feasible (even optimal) solution, when there are not taken into account sharply contrasting expert opinions/evaluations can lead to distortion of the final expertise assessment, where a measure of distortion remains unvalued and, moreover, there also remains unexplored the potential impact of this measure on the final assessment of examination. Consequently, this approach does not allow reducing the influence of distorted expert assessments on the final solution of decision-maker (DM);
- *The concept of "pursuit of experts' representativeness of experts"*, when the numerical assessments (that also may be fractional) are brought together without regard to the consistency of expert' opinion. As it has been proved in (Guseynov & Berezhnoy, 2011), this approach is not allowing to minimize the impact of biased (either due to the lack of qualifications of the experts or intentionally distorted) expert assessments.

Therefore, there is a need to construct a mathematical model that would allow minimizing the consequences of the lack of "traditional" models, based on the abovementioned two concepts. In (Guseynov & Berezhnoy, 2011), by the authors of this paper there was constructed the mathematical model for the analysis of expert information in order to determine the true ratings of the students (students or secondary school scholars) on the assembly of experts' evaluations. In this paper it is investigated the stability of some "traditional" models for analysis of expert' assessments. There are considered and analyzed traditional approaches to selection of the optimal regularization parameter in Tikhonov regularization method; there are identified its major shortcomings, and there is proposed a radically new approach for choosing the optimal or quasi-optimal regularization parameter.

QUALITATIVE AND MATHEMATICAL MODELS OF THE CONSIDERED PROBLEM

Generation of data

Let us assume that there are m students $\{s_1, s_2, ..., s_m\} \overset{def}{\equiv} S$, which are evaluated using n base characteristics / indicators / attributes $\{c_1, c_2, ..., c_n\} \overset{def}{\equiv} C$. Let us suppose that each of the students s_i $\left(i = \overline{1,m}\right)$ is somehow evaluated / "measured" using quantitative characteristics $\{a_{i,1}, a_{i,2}, ..., a_{i,n}\} \overset{def}{\equiv} A_{i\bullet}$, i.e. there exists an initial data matrix $\{a_{i,j}\}_{i=\overline{1,m}}^{j=\overline{1,n}} \overset{def}{\equiv} A \in \mathbb{R}^{m \times n}$, which will be called an estimation-identification matrix. In that way, $a_{i,j}$ element of the calculation-identification matrix A means value of the $j-$th $\left(j = \overline{1,n}\right)$ characteristic / attribute $c_j \in C$ $i-$th $\left(i = \overline{1,m}\right)$ of the student $s_i \in S$. In other words, a row-vector $A_{i\bullet} \in \mathbb{R}^n$ of the estimation-identification matrix A describes $i-$th student $s_i \in S$, and a column-vector $A_{\bullet j} \overset{def}{\equiv} \{a_{1,j}, a_{2,j}, ..., a_{m,j}\} \in \mathbb{R}^m$ of the estimation-identification matrix A contains assessments of the $j-$th characteristic / attribute $c_j \in C$ for all the students S evaluated by experts.

Construction of the integral indicator

Let us assume that each student $s_i \in S$ is attributed to the integrated indicator (integrated indicator is a convolution of data that compiled using special methods, which is the most informative to disclose a student in the space of characteristics / indicators / attributes; to obtain an exhaustive knowledge on integral indicators it is possible to refer to, for example, (Shakin, 1972; Ayvazyan, 2000; Hagerty et al., 2001; Ayvazyan & Mhitaryan, 1998; Borodkin & Ayvazyan, 2006; Orlov, 1996), where various methods are considered – "supervised" or "unsupervised" – in order to compile and calculate values of integral indicators; to understand the "inomnipotence" of integral indicators usage can be accessed, for example, in (Ayvazyan & Isakin, 2006; Strizhov, 2011)

$$I_i \overset{def}{\equiv} \sum_{j=1}^{n} w_j \cdot f_j^{Uniform\_scale}\left(a_{i,j}\right) \in \mathbb{R}^1, \quad \left(\forall i = \overline{1,m}\right) \tag{1}$$

where $w_j \in \mathbb{R}^1$ $\left(j = \overline{1,n}\right)$ is weight / importance of the $j-$th $\left(j = \overline{1,n}\right)$ characteristic / indicator / attribute c_j $\left(j = \overline{1,n}\right)$; function $f_j^{Uniform\_scale}\left(a_{i,j}\right)$ is a function of reduction of characteristics / indicators / attributes into the single scale, and is defined as

$$f_j^{Uniform\_scale} : a_{i,j} \mapsto (-1)^{Modifier_j} \cdot \frac{a_{i,j} - \min\limits_{i=\overline{1,m}}\{a_{i,j}\}}{\max\limits_{i=\overline{1,m}}\{a_{i,j}\} - \min\limits_{i=\overline{1,m}}\{a_{i,j}\}} + Modifier_j; \tag{2}$$

numerical parameter $Modifier_j$ is a modifier of single scale, and is determined as

$$Modifier_j = \begin{cases} 1, & if \quad \underset{i=\overline{1,m}}{optimal}\{a_{i,j}\} = \min\limits_{i=\overline{1,m}}\{a_{i,j}\}, \\ 0, & if \quad \underset{i=\overline{1,m}}{optimal}\{a_{i,j}\} = \min\limits_{i=\overline{1,m}}\{a_{i,j}\}. \end{cases} \tag{3}$$

Note the following property, which may take place in the expression (2): for some indices $\overline{j} \in \{1, ..., n\}$ the denominator of fraction (2) may be zero, and in this case, the corresponding characteristics $c_{\overline{j}} \in C$ (hence, the values of this characteristic $a_{i,\overline{j}} \in A_{\bullet\overline{j}}$, i.e. items) are excluded from the further consideration.

Further, from (1)-(3) there are calculated integral indicators (the methods for finding the valuesof integral indicators will be discussed later), and it is necessary to satisfy in some way the condition of compatibility of various integral indicators found for various students normalizing the general characteristics. To do this, the maximum and minimum possible values of each characteristic are set for the whole manifold of S students, and the optimal value of this characteristic is assigned. Then it becomes possible to rank the students: the best one is the student who possesses all characteristics having optimal values; the worst is the student who possesses all characteristics having worst values; characteristics of other students will be distributed (after applying the valuation map to characteristics) on the scale between the integral indicators of the best and the worst students, thus, providing the possibility to compare students with each other: (a) student $s_{\bar{i}} \in S$, which has the maximal value of integral indicator $I_{\bar{i}} = \max\limits_{i=\overline{1,m}} \{I_i\}$, is regarded to be the best; (b) student $s_{\bar{\bar{i}}} \in S$, which has the minimal value of integral indicator $I_{\bar{\bar{i}}} = \min\limits_{i=\overline{1,m}} \{I_i\}$, is regarded to be the worst; (c) characteristic / indicator / attribute $c_{\bar{i}} \in C$, which has a maximal value of weight $w_{\bar{i}}$, is the most significant while finding the integral indicator; (d) characteristic / indicator / attribute $c_{\bar{\bar{i}}} \in C$, which has minimal value of weight $w_{\bar{\bar{i}}}$, is the least important while finding the integral indicator.

Thus, (1)-(3) gives us vector-integral indicators of the manifold S, which is expressed in the form of operator equation

$$I = AW, \tag{4}$$

where $I \overset{def}{\equiv} \{I_1,...,I_m\} \in \mathbb{R}^m$; $W \overset{def}{\equiv} \{w_1,...,w_n\} \in \mathbb{R}^n$ is a manifold of significance / weights of characteristics/ indicators / attributes C.

Obviously, in order to determine the values of integral indicator I, subordinated to the equation (4), as the first step it is required to find somehow significance / weights of all indicators. In order to do this there exist various methods, which are divided (for instance, see (Ayvazyan, 2000; Ayvazyan & Mhitaryan, 1998; Borodkin & Ayvazyan, 2006; Orlov,1996; Strizhov, 2011) and respective references given in these) into "supervised" and "unsupervised" methods. Among these methods of both types there should be mentioned the following well-researched methods: metric method (finding the distance), principal components analysis, Pareto stratification method, weighted sum method, singular decomposition method, expert-statistical method, method of expert' evaluations correction in the linear scales, method of experts' evaluations in the ranked scales. In this work we will use the last of these methods – the method of correction of experts' evaluations in the ranked scales. A short essence of this method is as follows: experts put the grades of quality characteristics / indicators to students, as well as evaluate the significance of these characteristics / indicators in the ranked scale, suggesting the ratings linear order is assigned to the assessment multifold (thus, this approach is clearly based on ideas of experts' evaluations correction method in the linear scales); then the experts are given opportunity to evaluate the weights of general characteristics and integral indicators of students; one of the main tasks of experts is to identify contradictions / adjust differences between the integral indicators, weights of characteristics / indicators, weights of characteristics / indicators of students and the measured data of the students (such a contradiction arises if integral indicators are not consistent, which requires a special procedure). Some information about experts: the role of expert evaluation is quite essential for the considered problem, namely, let us assume that experts are setting the criteria by which students are assessed; students are clustered by experts based on these criteria; experts put grades to each of the students. Experts should comply with requirement that each expert should have his/her own professional opinion (not only based on the calculated / measured data but based on the personal experience and knowledge acquired prior to and in the process of work), not imposed by public opinion; experts should be free in their actions (for example, working with a special questionnaires; expressing their opinion in those questionnaires; putting comments to those questionnaires; etc.), in their utterances, etc.

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Thus, the result of operation of one expert is a triple (I_{ex}, W_{ex}, A). It should be noted that if there is a group of experts evaluating the quality characteristics / indicators of students and the weights of these characteristics/ indicators in the ranked scales, then they should be reduced to the agreed form, for example, by calculating the Kemeny median (for instance, see (Litvak, 1981), as well as (Guseynov & Berezhnoy, 2011)). The consistent values of integral indicators and weights of characteristics / indicators / attributes are such vectors I^* and W^* that satisfy stated conditions

$$I^* = AW^*; \quad W^* = A^{-1}I^*, \tag{5}$$

where with A^{-1} is denoted the operator inverse to A (in our case finite-dimensional, i.e. A is a matrix, however all the following statements remain fair in the case, when A operator is infinite-dimensional linearly bounded operator, for example, linear completely continuous operator, which has following properties $A^{-1}AA = A,\ \ A^{-1}AA^{-1} = A^{-1},\ \ (A^{-1}A)^T = A^{-1}A,\ \ (AA^{-1})^T = AA^{-1};$ I^* represent expert quality evaluations of students; W^* represent expert evaluations of students characteristics / indicators weights.

Remark 1. *As it has been already mentioned earlier in this section, in order to estimate weights / importance of the students' characteristics / indicators we use the experts' evaluations correction method in the ranked scales. However, it should be noted that the same conditions (5) (of course, in various interpretations and meanings) arise also in the application of principal components method, singular decomposition method, weighted sum method, expert-statistical method, as well as experts' evaluation correction method in the linear scales. Therefore, for the development and study of a regularizing algorithm, which will be discussed in next section of this paper, the obtained results are correct, if in order to assess students' characteristics / indicators weights there will be applied one of the above listed methods instead of the experts' evaluations correction method in ranked scales.*

CONSTRUCTION OF THE REGULARIZING ALGORITHM FOR THE STABLE INVERTED TRANSFORMATION OF THE INITIAL DATA MATRIX

Classical approach and arising problems

Let us consider the equation (5), where it is necessary to invert estimation-identification matrix A: $W^* = A^{-1}I^*$. As a rule the estimation-identification matrix A is an ill-conditioned matrix, hence, the inverse matrix A^{-1} doesn't exist in its classical meaning, i.e. the problem of finding weight evaluations for the students characteristics / indicators is becoming an ill-posed problem (for instance, see (Engl & Neubauer, 1985; Bauer & Lukas, 2011; Kojdecki, 1996; Kojdecki, 2000; Kojdecki, 2001; Sizikov, 2003; Ramm, 2005; Morozov, 1984; Verlan & Sizikov, 1986)): even small perturbation of experts' evaluation vector causes the substantial change of integral indicators vector. Therefore, there arises the problem of its stable inverse transformation (not finding of its pseudoinversion: the most widely known type of matrix pseudoinverse is the Moore-Penrose pseudoinverse. However this method appears to be incapable while inverting the estimation-identification matrix A) it is necessary to develop special methods that are using mathematical toolset of the theory of ill-posed problems. This issue becomes even more acute and complex when elements of the estimation-identification matrix A are characterized by errors, which maximum possible dispersion range is priory known. It is obvious that in this case each element of the estimation-identification matrix is defined not by a single number but in the interval way, i.e. instead of a_{ij} $\left(i=\overline{1,m};\ j=\overline{1,n}\right)$ there exists $\tilde{a}_{ij}$ $\left(i=\overline{1,m};\ j=\overline{1,n}\right)$, which may take any value from the interval $\left[a_{ij}-\delta_{ij},\ a_{ij}+\delta_{ij}\right]$ $\left(i=\overline{1,m};\ j=\overline{1,n}\right)$. Then, apart from the abovementioned problem of a stable inverse transformation of the estimation-identification matrix, there appears one more problem, namely, due to the fact that there exists not really a single ill-conditioned matrix A, but a whole family (containing, possibly infinitely large number of elements)of ill-conditioned matrices

$A(\delta)$, then it is important to find which of them is more or less adequately describes the estimation-identification values of students characteristics / indicators in order to implement the inverse transformation procedure exactly for that matrix.

In the next subsection of this paper it is proposed new regularizing algorithm required for the stable inverse transformation of matrix A. The background of the proposed algorithm contains the idea of Tikhonov regularization method (for instance, see (Engl & Neubauer, 1985; Bauer & Lukas, 2011; Kojdecki, 1996; Kojdecki, 2000; Kojdecki, 2001; Sizikov, 2003; Ramm, 2005; Morozov, 1984; Verlan & Sizikov, 1986)). However, as it will be obvious from the contents of next subsection of this paper, the proposed approach has a radical distinction from the Tikhonov classical approach to selection of the regularization parameter α in the Tikhonov functional $M^{\alpha}[z,u_{\delta}]$.

For the stable inverse transformation of estimation-identification matrix A let us consider the following operator equation:

$$Az = u, \tag{6}$$

where $z \in Z$ is the required element; $u \in U$ is a given element; Z and U are the Hilbert spaces; operator $A: Z \to U$ is a given linear bounded operator. Let us emphasize once again that operator in our specific case is finite-dimensional operator, i.e. it is a matrix, however all results that are obtained below remain valid also for infinite-dimensional linear bounded operators, in particular, for the completely continuous operators.

As it has been mentioned in subsection 2.1, elements of the estimation-identification matrix A are the result of assessment / "measurement" of students' characteristics / indicators and, hence, they are given with some errors. Therefore, instead of equation (6) we shall consider the approximate equation

$$A^{\{h\}} z = u^{\{\delta\}}, \tag{7}$$

where

$$\left\| A - A^{\{h\}} \right\| \le h,\ \left\| u - u^{\{\delta\}} \right\|_U \le \delta,\ \delta > 0,\ h \ge 0. \tag{8}$$

Denoted $\Delta \overset{def}{\equiv} (\delta; h)$, we able to formulate our target in the following way: it is necessary to find such solution $z^{\{\Delta\}} \in Z$ of the equation (7) based on given $\left\{A^{\{h\}}, u^{\{\delta\}}; \Delta\right\}$, which satisfy conditions (8) making it stable, i.e. that satisfies the following condition $\left\| z^{\{normal\}} - z^{\{\Delta\}} \right\|_Z \xrightarrow{\Delta \to 0} 0$, whereas $z^{\{normal\}} \in Z$ it is designated the normal pseudosolution (i.e. solution having a minimal norm function in the space Z) of the equation (6). In order to demonstrate a significant difference of a new approach proposed in this paper devoted to the selection of the optimal regularization parameter α in the Tikhonov functional $M^{\alpha}[z,u_{\delta}]$ from the traditional regularizing methods, let us briefly discuss the method of Tikhonov regularization (for instance, see (Engl & Neubauer, 1985; Bauer & Lukas, 2011; Kojdecki, 1996; Kojdecki, 2000; Kojdecki, 2001; Sizikov, 2003; Ramm, 2005; Morozov, 1984; Verlan & Sizikov, 1986)), focusing the particular attention on the problem of optimal regularization parameter selection taking the advantage of traditional methods (for instance, see (Engl & Neubauer, 1985; Bauer & Lukas, 2011; Kojdecki, 1996; Kojdecki, 2000; Kojdecki, 2001; Sizikov, 2003; Ramm, 2005; Morozov, 1984; Verlan & Sizikov, 1986; Guseynov & Volodko, 2003; Guseynov & Okruzhnova, 2005; Dmitriev & Guseynov, 1995; Guseynov, 2003) and respective references given in these).

In the Tikhonov regularization method instead of equation (7) there is considered and solved equation

$$\left(A^{\{h\}}\right)^* A^{\{h\}} z^{\alpha} + \alpha \cdot z^{\alpha} = \left(A^{\{h\}}\right)^* u^{\{\delta\}}, \tag{9}$$

where $\alpha = \alpha(\Delta) > 0$ is a regularization parameter; A^* is a conjugate to A operator. In (Engl & Neubauer, 1985; Bauer & Lukas, 2011; Kojdecki, 1996; Kojdecki, 2000; Kojdecki, 2001; Sizikov, 2003; Ramm, 2005; Morozov, 1984; Verlan & Sizikov, 1986; Guseynov & Volodko, 2003; Guseynov & Okruzhnova, 2005; Dmitriev & Guseynov, 1995; Guseynov, 2003) and in variety of corresponding papers there are presented diverse methods distinct on the classification and degree of accuracy both for selection of optimal and / or quasi-optimal regularization parameter $\alpha = \alpha(\Delta)$, and for estimation of regularizing solution z^α error $\left\| z^{\{normal\}} - z^\alpha \right\|_\bullet$. In all these approaches the major requirement / condition is the requirement of vicinity $\left\| z^{\{normal\}} - z^\alpha \right\|_\bullet$ to $\left\| z^{\{normal\}} - z^{\alpha_{exact\_optimal}} \right\|_\bullet = \min_{\alpha(\Delta)>0} \left\| z^{\{normal\}} - z^{\alpha(\Delta)} \right\|_\bullet$ in the asymptotics at $\Delta = (h;\delta) \to 0$, and not for the finite $h \geq 0$ and $\delta > 0$. In other words, the traditional approaches for the finite h and δ ensure the sufficiently good results only for the initially modeled problems, which are specially selected for the demonstration purposes allowing to show the capabilities of one or another method in respect to selection optimal and / or quasi-optimal regularization parameter. As it is shown in (Verlan & Sizikov, 1986), even if for some of these specially chosen modeled problems traditional algorithms of optimal and / or quasi-optimal regularization parameters selection for the finite values (even for arbitrary small values) h and δ provide an increased value $\alpha(\Delta)$ in comparison with the exact optimal $\alpha_{exact\_optimal}(\Delta)$ and, hence, there take place minimum two distortions: 1) the assessment $\left\| z^{\{normal\}} - z^\alpha \right\|_\bullet$ is increased compared to the required assessment $\left\| z^{\{normal\}} - z^{\alpha_{exact\_optimal}} \right\|_\bullet$; 2) the resolvability of the Tikhonov regularization method is decreased (for instance, see (Guseynov & Okruzhnova, 2005; Dmitriev & Guseynov, 1995)) – it means that the solution obtained using traditional approaches $z^\alpha \in Z$ is in fact smoother compared to the required solution $z^{\alpha_{exact\_optimal}} \in Z$. Therefore, there is the only conclusion: for the finite values of h and δ without a priori significant additional qualitative and / or quantitative assumptions regarding the desired solution of equation (6) in a traditional way it is not possible to obtain the exact optimal solution for regularization parameter $\alpha_{exact\_optimal}(\Delta)$, if only the specially chosen modeled problems are not studied. As it could be noticed from the solution of numerous modeled problems (for instance, see (Verlan & Sizikov, 1986; Guseynov & Volodko, 2003; Guseynov & Okruzhnova, 2005)), the abovementioned two distortions take place as soon as relative errors of the principal operator in the right-hand side of equation (7) is greater than 1%, i.e. when $relative\_error\_h > 1\%$ and $relative\_error\_\delta > 1\%$.

Regularizing algorithm with empirical way of regularization parameter selection

So, let us consider the problem (7), (8), and introduce the following designations: $\overline{A}^{\{h\}} \overset{def}{\equiv} \left(A^{\{h\}}\right)^* A^{\{h\}}$; $\overline{u}^{\{h;\delta\}} \overset{def}{\equiv} \left(A^{\{h\}}\right)^* u^{\{\delta\}}$. Then Tikhonov equation (9) takes the form

$$\overline{A}^{\{h\}} z^\alpha + \alpha \cdot z^\alpha = \overline{u}^{\{h;\delta\}}. \qquad (10)$$

Further, comparing the initial approximate equation (7) with the equation (10), it could be noted that in classical Tikhonov regularization method the initial equation is, in fact, not equation (7), but the equation

$$\overline{A}^{\{h\}} z^\alpha = \overline{u}^{\{h;\delta\}}, \qquad (11)$$

i.e. the right-hand side $u^{\{\delta\}} \in U$ of the initial equation (7) is not explicitly included into the classical method of Tikhonov regularization, while at the same time in all the different variations of the residual method, including the generalized residual principle (for instance, see (Morozov, 1984)), it is used the right-hand side $u^{\{\delta\}} \in U$, error δ instead of the right-hand side error $\overline{u}^{\{h;\delta\}} \in U$, which, as it is obvious from (11), is dependent not only on δ, but also on h. Hence, random errors in $u^{\{\delta\}} \in U$ may be substantially smoothened and, therefore, relative error $\overline{u}^{\{h;\delta\}} \in U$, which, as it was recently mentioned, in classical Tikhonov regularization method is not taken into account, may be substantially different (even by several orders!) from the relative error $u^{\{\delta\}} \in U$, which, we have cleared out above, is the only one that is taken into account in the classical Tikhonov regularization method.

Remark 2. *Just noted fact becomes quite evident, if in the equation (7) as the principal operator* $A^{\{h\}}$ *to take, for example, the Fredholm integral operator, acting from* $L_2[a,b]$ *to* $L_2[a,b]$, *i.e.* $A^{\{h\}}[\bullet] \overset{def}{\equiv} \int_a^b K^{\{h\}}(x,y)\cdot[\bullet]dy$: *in this case, we have* $\bar{u}^{\{h;\delta\}}(x) = \int_a^b K^{\{h\}}(x,y)\cdot u^{\{\delta\}}(y)dy$ *and, therefore, due to the fact that integration operation is a smoothing filter, then with respect to function* $u^{\{\delta\}}(y) \in L_2[a,b]$ *we obtain a sufficiently smoothed function* $\bar{u}^{\{h;\delta\}}(x) \in L_2[a,b]$.

Outlined in the Remark 2 property with smoothing of random errors in the equal measure refers to the right-hand side of equation (11), namely, in the left-hand side of this equation there is located a principal operator $\bar{A}^{\{h\}}$, which error could differ substantially from the error of the principal operator $A^{\{h\}}$ of the initial approximate equation (7), however in the classical Tikhonov regularization method it is taken into account particularly the error of operator $A^{\{h\}}$, instead of the error referring to the actual (i.e. that is really present in equation (6)) operator $\bar{A}^{\{h\}} = \left(A^{\{h\}}\right)^* A^{\{h\}}$.

To summarize the abovementioned, we conclude that it is necessary to use instead of errors $\Delta = (\delta;h)$ relevant to initial data $\left\{A^{\{h\}};u^{\{\delta\}}\right\}$ of the problem (7), (8) in a Tikhonov regularization method, errors $\bar{\Delta}$ of the initial data $\left\{\bar{A}^{\{h\}};\bar{u}^{\{h;\delta\}}\right\}$. Hence, we propose instead of the initial problem (7), (8) immediately consider equation (11), in addition taking into account in the Tikhonov regularization method errors of the actual initial data of that equation, i.e. principal operator $\bar{A}^{\{h\}} \overset{def}{\equiv} \left(A^{\{h\}}\right)^* A^{\{h\}}$ errors as well as errors of the element $\bar{u}^{\{h;\delta\}} \overset{def}{\equiv} \left(A^{\{h\}}\right)^* u^{\{\delta\}}$. This consideration, as it will be seen from the hereinafter contained treatment causes the fundamental difference in optimal and / or quasi-optimal regularization parameter selection. Let us present this methodology. In accordance with generalized residual principle (for instance, see (Morozov, 1984) as well as (Bauer & Lukas, 2011; Kojdecki, 1996)), regularization parameter $\alpha = \alpha(\Delta) > 0$ is a root of the equation

$$\left\|A^{\{h\}}z^\alpha - u^{\{\delta\}}\right\|_U = \left(\delta + h\cdot\left\|z^\alpha\right\|_Z\right)^2 + \left(\inf_{z\in Z}\left\|A^{\{h\}}z - u^{\{\delta\}}\right\|_U\right)^2, \tag{12}$$

where $\inf_{z\in Z}\left\|A^{\{h\}}z - u^{\{\delta\}}\right\|_U$ is a measure of incompatibility of the initial problem(7), (8). As it is shown in (Kojdecki, 1996) (as well as see (Kojdecki, 2000; Kojdecki, 2001)), regularization parameter $\alpha = \alpha(\Delta) > 0$ is the root of the equation

$\alpha^k \cdot \left\|\left(A^{\{h\}}\right)^* A^{\{h\}}z^\alpha - \left(A^{\{h\}}\right)^* u^{\{\delta\}}\right\|_U = \lambda\cdot\left\|A^{\{h\}}\right\|\cdot\left(\delta + h\cdot\left\|z^\alpha\right\|_Z\right)$, where $k \geq 0$ and $\lambda > 0$ are some constants.

From here, in respect to (9), we obtain

$$\alpha^{k+1}\cdot\left\|z^\alpha\right\|_Z = \lambda\cdot\left\|A^{\{h\}}\right\|\cdot\left(\delta + h\cdot\left\|z^\alpha\right\|_Z\right). \tag{13}$$

In this paper for finding the optimal regularization parameter it is proposed to use instead of equation (13), the following equation, whose root is a desired regularization parameter (this statement will be revealed and mathematically strictly proved):

$$\left\|z^{\alpha}\right\|_{Z}\cdot\left(\alpha^{k+1}-\lambda\cdot\sup_{h\ge 0}\left\|\bar{A}-\bar{A}^{\{h\}}\right\|\right)=\lambda\cdot\sup_{h\ge 0,\delta>0}\left\|\bar{u}-\bar{u}^{\{h;\delta\}}\right\|_{U}, \tag{14}$$

where $k\ge 0;\ \ \lambda>0;\ \bar{A}\overset{def}{\equiv}A^{*}A;\ \ \bar{u}\overset{def}{\equiv}A^{*}u.$

Main difference of equations (13) and (14) is rooted in the fact that value of "new" variable $\Sigma_{New}(\alpha)\overset{def}{\equiv}\sup\limits_{h\ge 0,\delta>0}\left\|\bar{u}-\bar{u}^{\{h;\delta\}}\right\|_{U}+\left\|z^{\alpha}\right\|_{Z}\cdot\sup\limits_{h\ge 0}\left\|\bar{A}-\bar{A}^{\{h\}}\right\|$

In the equation (14) cannot exceed the value of "old" variable $\Sigma_{Old}(\alpha)\overset{def}{\equiv}\left\|A^{\{h\}}\right\|\cdot\left(\delta+h\cdot\left\|z^{\alpha}\right\|_{Z}\right)$ in the equation (13), i.e. $\Sigma_{New}(\alpha)\le\Sigma_{Old}(\alpha)\ \forall\alpha>0,$ moreover, this inequality can be very strict. What is it useful for? – This ensures that the root $\alpha_{optimal}\overset{def}{\equiv}\alpha_{root}>0$ of the equation (14) and residual $\left\|z^{\{normal\}}-z^{\alpha_{optimal}}\right\|_{Z}$ are not overstated. Furthermore, by comparing equations (13) and (14), we can see that that the idea of taking into account actual initial data errors from the equation (6) in the Tikhonov regularization method (i.e. taking into account error of principal operator $\bar{A}^{\{h\}}$ in the equation (11) and errors of the right-hand side of the element $\bar{u}^{\{h;\delta\}}$ of the equation (11)) ensures equality to zero of the incompatibility measure of the equation (11), i.e. $\inf\limits_{z\in Z}\left\|\bar{A}^{\{h\}}z-\bar{u}^{\{h;\delta\}}\right\|_{U}=0.$ This fact is significant, and it fundamentally distinguishes the proposed equation (14) and equation (12) of the generalized residual principle.

Asymptotical assessments and outcome of regularizing operator

Now let us ask the main question – whether the discovered root $\alpha_{root}>0$ of the proposed equation (14) induces the regularizing operator? If the answer is positive, then there arise two more questions: (a) under which conditions the solution of proposed equation (9) exist and is unique? (b) what is the order of the found regularization solution $z^{\alpha_{optimal}}=z^{\alpha_{root}}$ to the normal pseudosolution $z^{\{normal\}}$? In order to answer these important questions, first of all it is necessary to set some estimates both for the root $\alpha_{root}>0$ of the equation (14), and for the residual $\left\|z^{\{normal\}}-z^{\alpha_{root}}\right\|_{Z}$. It is easy to see that under the conditions

$$\begin{cases}\dfrac{\sup\limits_{h\ge 0,\delta>0}\left\|\bar{u}-\bar{u}^{\{h;\delta\}}\right\|_{U}}{\left\|\bar{u}^{\{h;\delta\}}\right\|_{U}}<\dfrac{1}{\lambda} & if\quad k=0;\\ \left\|\bar{u}^{\{h;\delta\}}\right\|_{U}\ne 0 & if\quad k>0\end{cases} \tag{15}$$

Function $\left\|z^{\alpha}\right\|_{Z}\cdot\alpha^{k+1}$ as a function depending from the argument α is a monotonically increasing continuous function on semiaxis $(0,+\infty)$, and function $\lambda\cdot\left(\left\|z^{\alpha}\right\|_{Z}\cdot\sup\limits_{h\ge 0}\left\|\bar{A}-\bar{A}^{\{h\}}\right\|+\sup\limits_{h\ge 0,\delta>0}\left\|\bar{u}-\bar{u}^{\{h;\delta\}}\right\|_{U}\right)$ as well as function dependent on argument α is a monotonically decreasing continuous function located on the same semiaxis. Moreover, under the conditions (15) there take place the following asymptotics:

$\left\|z^{\alpha}\right\|_{Z}\cdot\alpha^{k+1}\xrightarrow{\alpha\to+\infty}+\infty,$ if $k>0,\ \left\|\bar{u}^{\{h;\delta\}}\right\|_{U}\ne 0;$

$\left\|z^{\alpha}\right\|_{Z}\cdot\alpha^{k+1}\xrightarrow{\alpha\to+\infty}\left\|\bar{u}^{\{h;\delta\}}\right\|_{U},$ if $k=0;$

$\left\|z^{\alpha}\right\|_{Z}\cdot\alpha^{k+1}\xrightarrow{\alpha\to+\infty}0,$ if $k>0,\ \left\|\bar{u}^{\{h;\delta\}}\right\|_{U}=0;$

$\left\|z^{\alpha}\right\|_{Z}\cdot\alpha^{k+1}\xrightarrow{\alpha\to 0+0}0;$

$\left\|z^{\alpha}\right\|_{Z}\cdot\sup\limits_{h\ge 0}\left\|A^{*}A-\bar{A}^{\{h\}}\right\|+\sup\limits_{h\ge 0,\delta>0}\left\|A^{*}u-\bar{u}^{\{h;\delta\}}\right\|_{U}\xrightarrow{\alpha\to+\infty}\sup\limits_{h\ge 0,\delta>0}\left\|A^{*}u-\bar{u}^{\{h;\delta\}}\right\|_{U};$

$$\sup_{h\geq 0,\delta>0}\left\|A^{*}u-\bar{u}^{\{h;\delta\}}\right\|_{U} < \lim_{\alpha\to 0+0}\left(\left\|z^{\alpha}\right\|_{Z}\cdot\sup_{h\geq 0}\left\|A^{*}A-\bar{A}^{\{h\}}\right\|+\sup_{h\geq 0,\delta>0}\left\|A^{*}u-\bar{u}^{\{h;\delta\}}\right\|_{U}\right).$$

Further, since the proposed equation (14) is equivalent to the equation

$$\left\|z^{\alpha}\right\|_{Z}\alpha^{k+1}=\lambda\left(\left\|z^{\alpha}\right\|_{Z}\sup_{h\geq 0}\left\|\bar{A}-\bar{A}^{\{h\}}\right\|+\sup_{h\geq 0,\delta>0}\left\|\bar{u}-\bar{u}^{\{h;\delta\}}\right\|_{U}\right), \tag{16}$$

where $k\geq 0;\ \lambda>0,$ then given above asymptotic assessments, along with the abovementioned properties of strict monotonicity and continuity of the functions $\left\|z^{\alpha}\right\|_{Z}\cdot\alpha^{k+1}$ and $\lambda\cdot\left(\left\|z^{\alpha}\right\|_{Z}\cdot\sup_{h\geq 0}\left\|\bar{A}-\bar{A}^{\{h\}}\right\|+\sup_{h\geq 0,\delta>0}\left\|\bar{u}-\bar{u}^{\{h;\delta\}}\right\|_{U}\right),$ and which, as it is evident from (16), are respectively the left-hand and right-hand sides of this equation, allow ascertaining the following significant fact (now it is obvious by virtue of the Brouwer-Schauder Fixed Point Theorem: for instance, see (Hutson & Pym, 1980)): if the conditions (15) are satisfied, the equation (16) (hence, the equivalent to it equation (14)) has the only fixed point, i.e. equation (14) has the single root, and particularly this single root will be taken as optimal regularization parameter in the Tikhonov regularization method (i.e. in the Tikhonov equation (10)). Exhaustive answer to this above pointed question should be found since (a), having no answer it is impossible to find an answer to the main question (b) – is the unique equation (14) root being found inducing the regularizing operator? To answer this major question, first let us give some upper estimates, which correctness is fairly easy revealed, by using known facts, that for bounded linear operators A and B, reflecting the Banach space Z into the Banach space U, there are valid $\|A\|=\|A^{*}\|$ and $\|AB\|\leq\|A\|\cdot\|B\|$:

$$\begin{cases}\sup_{h\geq 0}\left\|\bar{A}-\bar{A}^{\{h\}}\right\|\leq 2\cdot\left\|A^{\{h\}}\right\|\cdot h;\\ \sup_{h\geq 0,\delta>0}\left\|\bar{u}-\bar{u}^{\{h;\delta\}}\right\|_{U}\leq\left\|A^{\{h\}}\right\|\cdot\delta+\left\|u^{\{\delta\}}\right\|_{U}\cdot h.\end{cases} \tag{17}$$

Further let us assume

$$\frac{\sup_{h\geq 0,\delta>0}\left\|\bar{u}-\bar{u}^{\{h;\delta\}}\right\|_{U}}{\left\|\bar{u}^{\{h;\delta\}}\right\|_{U}}\leq\frac{\left\|\bar{A}^{\{h\}}\right\|^{k}}{2\cdot\lambda}-\frac{\sup_{h\geq 0}\left\|\bar{A}-\bar{A}^{\{h\}}\right\|}{\left\|\bar{A}^{\{h\}}\right\|}, \tag{18}$$

which, in fact, is generalization of the first inequality in (15). Satisfying the (18) condition guarantees the validity of the following very useful (especially when the finite errors $\Delta=(h;\delta)$ of the initial data of the initial problem (7), (8)) upper estimate for the root (and as it has been proved above, unique) $\alpha_{root}>0$ of the equation (14):

$$\alpha_{root}^{k+1}\leq\left(\sup_{h\geq 0,\delta>0}\left\|\bar{u}-\bar{u}^{\{h;\delta\}}\right\|_{U}+\sup_{h\geq 0}\left\|\bar{A}-\bar{A}^{\{h\}}\right\|\right)\cdot\left(\lambda\cdot\max\left\{1,\frac{2\cdot\left\|\bar{A}^{\{h\}}\right\|}{\left\|\bar{u}^{\{h;\delta\}}\right\|_{U}}\right\}\right), \tag{19}$$

From which at $\sup_{h\geq 0}\left\|\bar{A}-\bar{A}^{\{h\}}\right\|\to 0,\ \sup_{h\geq 0,\delta>0}\left\|\bar{u}-\bar{u}^{\{h;\delta\}}\right\|_{U}\to 0$ there immediately and directly follows the asymptotic assessment $\alpha_{root}=O\left(\sup_{h\geq 0,\delta>0}\left\|\bar{u}-\bar{u}^{\{h;\delta\}}\right\|_{U}+\sup_{h\geq 0}\left\|\bar{A}-\bar{A}^{\{h\}}\right\|\right)^{\frac{1}{k+1}}$, which is regarded to be less useful (i.e. rougher, allowing the over the optimal value of the optimal regularization parameter) in solving the real problem of determining the diagnostic matrix for finding a stable solution of estimate-identification parameters of the dual-circuit gas turbine engine. Now having at our disposal the obtained above results (namely, having equation (14); upper estimates (17) for errors of principal operator and for the right-hands side of the equation (11); conditions (15) and (18); upper estimate (19) for the root of the equation (14)), we could give the upper evaluation to the error of the solution $z^{\alpha_{root}}$ of the Tikhonov equation (10), where selection of optimal and/or quasi-optimal regularization parameter is achieved by solving of the proposed and justified equation (14) instead of traditional approaches. Let us estimate the residual $\left\| z^{\{normal\}} - z^{\alpha_{root}} \right\|_Z \equiv \left\| z^{\{normal\}} - z^{\alpha_{optimal}} \right\|_Z$. In order to do that, alongside with the obtained results, let us apply the following results from (Kojdecki, 1996; Kojdecki, 2000; Kojdecki, 2001):

$$\left\| z^{\{normal\}} - z^{\alpha} \right\|_Z \le \alpha^{-1} \left(\sup_{h \ge 0} \left\| A^* A - \bar{A}^{\{h\}} \right\| + \sup_{h\ge 0, \delta > 0} \left\| A^* u - \bar{u}^{\{h;\delta\}} \right\|_U \right) \max\left\{1, \left\| z^{\{normal\}} \right\|_Z \right\} + \alpha \left\| \left(\alpha E + \left(A^{\{h\}}\right)^* A^{\{h\}} \right)^{-1} z^{\{normal\}} \right\|_U, \quad (20)$$

whereas E is denoted the unit operator. In (Kojdecki, 1996; Kojdecki, 2000; Kojdecki, 2001) the following three useful evaluations were achieved, which imply the boundedness of the operator $\left(\alpha \cdot E + \left(A^{\{h\}}\right)^* A^{\{h\}} \right)^{-1}$:

$$\left\| \left(\alpha \cdot E + \left(A^{\{h\}}\right)^* A^{\{h\}} \right)^{-1} \left(A^{\{h\}}\right)^* A^{\{h\}} \right\|_U \le 1;$$

$$\left\| \left(\alpha \cdot E + \left(A^{\{h\}}\right)^* A^{\{h\}} \right)^{-1} \left(A^{\{h\}}\right)^* \right\|_U \le \frac{1}{2 \cdot \sqrt{\alpha}};$$

$$\left\| \left(\alpha \cdot E + \left(A^{\{h\}}\right)^* A^{\{h\}} \right)^{-1} \right\|_U \le \frac{1}{\alpha},$$

Let us note that first two evaluations may be successfully applied to the residual (20). Along with the obtained results we will also use the estimate (20). For now our main objective is to identify the upper bound of the norm function $\left\| \left(A^{\{h\}}\right)^* A^{\{h\}} \left(z^{\{normal\}} - z^{\alpha_{root}} \right) \right\|_U$:

$$\left\| \left(A^{\{h\}}\right)^* A^{\{h\}} \left(z^{\{normal\}} - z^{\alpha_{root}} \right) \right\|_U = \left\| \left(A^{\{h\}}\right)^* A^{\{h\}} \left\{ \left(\alpha_{root} \cdot E + \left(A^{\{h\}}\right)^* A^{\{h\}} \right)^{-1} \left(A^{\{h\}}\right)^* u^{\{\delta\}} \right.\right.$$

$$\left.\left. - \left(\alpha_{root} \cdot E + \left(A^{\{h\}}\right)^* A^{\{h\}} \right)^{-1} \left(\alpha_{root} \cdot E + \left(A^{\{h\}}\right)^* A^{\{h\}} \right) z^{\alpha_{root}} \right\} \right\|_U = \left\| \left(A^{\{h\}}\right)^* A^{\{h\}} \left(\alpha_{root} \cdot E + \left(A^{\{h\}}\right)^* A^{\{h\}} \right)^{-1} \left\{ \left(A^{\{h\}}\right)^* u^{\{\delta\}} - A^* u \right\} \right.$$

$$\left. + \left(A^{\{h\}}\right)^* A^{\{h\}} \left(\alpha_{root} \cdot E + \left(A^{\{h\}}\right)^* A^{\{h\}} \right)^{-1} \left\{ A^* A - \left(A^{\{h\}}\right)^* A^{\{h\}} \right\} z^{\alpha_{root}} - \alpha_{root} \cdot \left(A^{\{h\}}\right)^* A^{\{h\}} \left(\alpha_{root} \cdot E + \left(A^{\{h\}}\right)^* A^{\{h\}} \right)^{-1} z^{\alpha_{root}} \right\|_U$$

$$\le \left(\sup_{h \ge 0} \left\| A^* A - \bar{A}^{\{h\}} \right\| + \sup_{h\ge 0, \delta > 0} \left\| A^* u - \bar{u}^{\{h;\delta\}} \right\|_U \right) \cdot \max\left\{1, \left\| z^{\{normal\}} \right\|_Z \right\} + \alpha_{root} \cdot \left\| z^{\alpha_{root}} \right\|_Z.$$

The following upper bound has been found:

$$\left\| \left(A^{\{h\}}\right)^* A^{\{h\}} \left(z^{\{normal\}} - z^{\alpha_{root}} \right) \right\|_U \le \left(\sup_{h \ge 0} \left\| A^* A - \bar{A}^{\{h\}} \right\| + \sup_{h\ge 0, \delta > 0} \left\| A^* u - \bar{u}^{\{h;\delta\}} \right\|_U \right) \max\left\{1, \left\| z^{\{normal\}} \right\|_Z \right\} + \alpha_{root} \left\| z^{\alpha_{root}} \right\|_Z. \quad (21)$$

Obtained in equality (21) allows positively answering to the above stated main question: proposed and justified empirical choice of the optimal regularization parameter $\alpha_{optimal}$ as a solution α_{root} of equation (14) induces Tikhonov regularizing operator. Really, taking into account inequalities (17) in the newly obtained upper bound (21), and then passing in the resulting inequality to the limit at $\Delta = (h;\delta) \to 0$, we have $\left\| z^{\{normal\}} - z^{\alpha_{root}} \right\|_Z = O(\delta + h)^{\frac{1}{k+1}}$, that proves the convergence by norm function (i.e. strong convergence!) of the normalized solution $z^{\alpha_{optimal}} \equiv z^{\alpha_{root}}$, obtained from the Tikhonov equation (10), where as an optimal regularization parameter $\alpha_{optimal}$ it was taken the single root of the equation (14), to the normal pseudosolution $z^{\{normal\}}$. Moreover, for the error of resulting regularized solution there is present an upper estimate (21), and for the optimal regularization parameter $z^{\alpha_{optimal}} \equiv z^{\alpha_{root}}$ is valid the upper estimate (19).

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